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HIGHER-DIMENSIONAL CROSSED MODULES AND THE HOMOTOPY GROUPS OF $(n+1)$-ADS

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Introduction

In this paper (Theorem 3.7) we compute the low-dimensional homotopy groups of certain $(n+1)$-ads $(X; X_1, \dots, X_n)$ for which $X_1, \dots, X_n$ forms an open cover of X. We use this in Theorem 3.8 to generalise the result of Barratt and Whitehead [1, 5.3] on $(n+1)$-ads $(X; X_1, \dots, X_n)$ for which

$$X = \bigcup_{i=1}^{n} X_{\{i\}},$$

where $X_{\{i\}}$ denotes $\bigcap_{j \neq i} X_j$. They show that if $C = \bigcap_{j=1}^{n} X_j$ is simply connected, the $(X_{\{i\}}, C)$ are q_i-connected ($q_i \geq 2$), and $Q = q_1 + \cdots + q_n$, then $\pi_m(X; X_1, \dots, X_n) = 0$ for $m \leq Q$ and

$$\pi_{Q+1}(X; X_1, \dots, X_n) \cong \bigoplus^{(n-1)!} \bigotimes_{i=1}^{n} \pi_{q_i+1}(X_{\{i\}}, C).$$

We show here that C need only be path-connected, and we also get results when some of the q_i are 1 (so that the groups involved need not be abelian).

For $n=2$ the results generalise the classical Blakers–Massey theorem, and are due to Brown and Loday [2–4]; we follow their methods. These involve cat^n-groups (or n-cat-groups) as defined in [8, 1.2]. In fact it is convenient to replace cat^n-groups by more combinatorial algebraic systems which we call crossed n-cubes; a crossed 1-cube is a crossed module, and the crossed n-cubes are the higher-dimensional crossed modules of the title.

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The equivalence of cat^n-groups and crossed n-cubes is demonstrated in Section 1. In the course of it (Theorem 1.2) we solve the following group-theoretic problem (partly solved by End [6, Theorem D]): given groups M_A ($A \subset \{1, \ldots, n\}$) and functions $h: M_A \times M_B \to M_{A \cup B}$ ($A, B \subset \{1, \ldots, n\}$) find necessary and sufficient conditions for there to be a group structure on the set-theoretic Cartesian product $\prod M_A$ such that the h are restrictions of the commutator.

We make use of particular kinds of crossed n-cubes ('r-universal'); these are studied in Section 2. The applications to $(n+1)$-ads can be found in Section 3.

Throughout the paper n is a non-negative integer and $\langle n \rangle = \{1, \ldots, n\}$. If A is a set, then $|A|$ denotes its cardinality. For a and b elements of a group, $[a, b]$ is the commutator $aba^{-1}b^{-1}$. All spaces have base-points and all maps are base-point preserving.

The present paper improves on the description of crossed n-cubes in [5].

1. Crossed n-cubes

Recall from [8] that a *cat^n-group* (called there an *n-cat-group*) is a group G together with $2n$ endomorphisms s_i, t_i ($1 \le i \le n$) such that

$$t_i s_i = s_i, \quad s_i t_i = t_i, \quad [\operatorname{Ker} s_i, \operatorname{Ker} t_i] = 1 \qquad \text{for all } i,$$
$$s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i \qquad \text{for } i \ne j.$$

From [8, 2.2, 5.2] a cat^1-group is equivalent to a crossed module and a cat^2-group to a crossed square. (The definition of a crossed square is due to Guin–Walery and Loday [7].) We shall show that in general a cat^n-group is equivalent to a crossed n-cube as defined below. Recall that $\langle n \rangle$ is $\{1, \ldots, n\}$.

Definition. A *crossed n-cube* is a family of groups M_A ($A \subset \langle n \rangle$) together with homomorphisms $\mu_i : M_A \to M_{A \setminus \{i\}}$ ($i \in \langle n \rangle, A \subset \langle n \rangle$) and functions $h: M_A \times M_B \to M_{A \cup B}$ ($A, B \subset \langle n \rangle$) such that if ${}^a b$ denotes $h(a, b)b$ for $a \in M_A$ and $b \in M_B$ with $A \subset B$, then for $a, a' \in M_A$, $b, b' \in M_B$, $c \in M_C$ and for $i, j \in \langle n \rangle$,

$$\mu_i a = a \quad \text{if } i \notin A, \tag{1}$$

$$\mu_i \mu_j a = \mu_j \mu_i a, \tag{2}$$

$$\mu_i h(a, b) = h(\mu_i a, \mu_i b), \tag{3}$$

$$h(a, b) = h(\mu_i a, b) = h(a, \mu_i b) \quad \text{if } i \in A \text{ and } i \in B, \tag{4}$$

$$h(a, a') = [a, a'], \tag{5}$$

$$h(a, b) = h(b, a)^{-1}, \tag{6}$$

$$h(a, b) = 1 \quad \text{if } a = 1 \text{ or } b = 1, \tag{7}$$

$$h(aa', b) = {}^a h(a', b) h(a, b), \tag{8}$$

$$h(a,bb') = h(a,b)\,{}^b h(a,b'), \tag{9}$$

$$^a h(h(a^{-1},b),c)\,{}^c h(h(c^{-1},a),b)\,{}^b h(h(b^{-1},c),a) = 1, \tag{10}$$

$$^a h(b,c) = h({}^a b, {}^a c) \quad \text{if } A \subset B \text{ and } A \subset C. \tag{11}$$

A *morphism of crossed n-cubes* $(M_A) \to (M'_A)$ is a family of homomorphisms $M_A \to M'_A$ $(A \subset \langle n \rangle)$ which commute with the maps μ_i and the functions h.

Note that (1)–(2) amount to saying that the M_A and the μ_i form a commutative n-cube of groups, or, more formally, a functor on the partially ordered set of subsets of $\langle n \rangle$. Note also that (6)–(11) are standard commutator relations. One may therefore get an example of a crossed n-cube as follows: $M_{\{1\}}, \dots, M_{\{n\}}$ are normal subgroups of a group $M_\emptyset$, $M_A = \bigcap_{i \in A} M_{\{i\}}$ for $A \neq \emptyset$, the μ_i are inclusions and the h are restrictions of the commutator in $M_\emptyset$.

Note also that if $A \subset B$, then $(a,b) \mapsto {}^a b : M_A \times M_B \to M_B$ defines an action of M_A on M_B, by (7)–(9).

In view of (1) and (4)–(6), many of the μ_i and h are redundant. For $n=1$ the definition can be simplified considerably, and one finds that it reduces to that of a crossed module (see for example [8, 2.1]). For general n, however, the conditions are easier to state if one keeps the redundant functions.

We shall show that a cat^n-group is equivalent to a crossed n-cube. We use two results concerning a simpler situation, in which the μ_i are dropped. Note that in a cat^n-group the s_i are *projections* (that is, $s_i s_i = s_i$), since

$$s_i s_i = s_i t_i s_i = t_i s_i = s_i.$$

Our simplification of a cat^n-group will be a group G with n commuting projections $s_1, \dots, s_n$, as studied in [6] (though we do not require $s_1 \cdots s_n$ to be trivial).

Theorem 1.1. *Let G be a group with n commuting projections $s_1, \dots, s_n$. For $A \subset \langle n \rangle$ let*

$$M_A = \bigcap_{i \in A} \operatorname{Ker} s_i \cap \bigcap_{i \notin A} \operatorname{Im} s_i$$

and for $A, B \subset \langle n \rangle$ let $h : M_A \times M_B \to M_{A \cup B}$ be the restriction of the commutator function in G. Then

(i) *for any total ordering of the subsets of $\langle n \rangle$ the multiplication function*

$$(x_A) \mapsto \prod_A x_A : \prod_{A \subset \langle n \rangle} M_A \to G$$

is bijective;

(ii) *G can be obtained from the free product of the M_A by imposing the relations $h(a,b) = [a,b]$ $(a \in M_A,\ b \in M_B,\ A, B \subset \langle n \rangle)$:*

$$G = \Big(\mathop{*}_{A \subset \langle n \rangle} M_A \Big) / (h(a,b) = [a,b]).$$

This is [6, Theorems A and B]; a proof will be given below. Note that the total ordering of the subsets of $\langle n\rangle$ in (i) is completely arbitrary and need have no connection with the partial ordering by inclusion. Note also that the bijection in (i) is not usually a homomorphism. For $n=1$ the theorem essentially says that G is a semi-direct product of $M_{\{1\}}$ and $M_\emptyset$.

Theorem 1.2. *Suppose that M_A ($A\subset\langle n\rangle$) is a family of groups and $h: M_A\times M_B\to M_{A\cup B}$ ($A,B\subset\langle n\rangle$) is a family of functions. Then the following statements are equivalent:*

(i) *There is a group G with n commuting projections $s_1,\dots,s_n$ such that*

$$M_A=\bigcap_{i\in A}\operatorname{Ker} s_i\cap\bigcap_{i\notin A}\operatorname{Im} s_i \quad \textit{for } A\subset\langle n\rangle$$

and the h are restrictions of the commutator function in G;

(ii) *Equations* (5)–(11) *hold.*

This generalises [6, Theorem D] and will be proved below. For $n=1$ it says that there is a semi-direct product of $M_\emptyset$ with $M_{\{1\}}$ if and only if $M_\emptyset$ acts on $M_{\{1\}}$.

Assuming Theorems 1.1 and 1.2 we can prove the main result of this section.

Theorem 1.3. *The categories of cat^n-groups and crossed n-cubes are equivalent.*

Proof. By Theorem 1.2 a group G with n commuting projections $s_1,\dots,s_n$ is equivalent to a pair of families (M_A,h) satisfying (5)–(11). The equivalence is given by

$$M_A=\bigcap_{i\in A}\operatorname{Ker} s_i\cap\bigcap_{i\notin A}\operatorname{Im} s_i,$$

$$h(a,b)=[a,b]\quad \text{for } a\in M_A,\ b\in M_B,$$

$$G=\Big(\mathop{*}_{A} M_A\Big)/(h(a,b)=[a,b]),$$

where the final statement comes from Theorem 1.1(ii). It suffices to show that endomorphisms t_i of G making (G,s_i,t_i) a cat^n-group are equivalent to homomorphisms $\mu_i: M_A\to M_{A\setminus\{i\}}$ satisfying (1)–(4).

Note that for $a\in M_A$,

$$s_i a=\begin{cases}1, & i\in A,\\ a, & i\notin A,\end{cases}$$

since $s_is_i=s_i$. It is now easy to check that an endomorphism t_i of G satisfying $s_it_i=t_i$ and $s_jt_i=t_is_j$ for $j\neq i$ is equivalent to homomorphisms $\mu_i: M_A\to M_{A\setminus\{i\}}$ satisfying (3); one takes the μ_i to be restrictions of t_i. It is also easy to check that $t_is_i=s_i$ is equivalent to (1) and $t_it_j=t_jt_i$ is equivalent to (2). It remains to show that $[\operatorname{Ker} s_i,\operatorname{Ker} t_i]=1$ is equivalent to (4).

For this we use Theorem 1.1(ii). For the fixed i under consideration, order the subsets of $\langle n\rangle$ so that if $i\in A$, then A immediately precedes $A\setminus\{i\}$. It is then apparent that $\operatorname{Ker} s_i$ is generated by the elements $a\in M_A$ $(i\in A)$ and $\operatorname{Ker} t_i$ is generated by the elements $b^{-1}(t_ib)$ $(b\in M_B, i\in B)$. So $[\operatorname{Ker} s_i, \operatorname{Ker} t_i]=1$ is equivalent to $[a, b^{-1}(t_ib)]=1$ for $a\in M_A$, $b\in M_B$, $i\in A$, $i\in B$. Since

$$[a, b^{-1}(t_ib)] = b^{-1}[a,b]^{-1}[a, t_ib]b,$$

this is equivalent to $h(a,b)=h(a,\mu_ib)$ for $a\in M_A$, $b\in M_B$, $i\in A$, $i\in B$, which is one half of (4). Since the other half $(h(a,b)=h(\mu_ia,b))$ is redundant in view of (6), this completes the proof. $\square$

Next we give a proof of Theorem 1.1(i) more direct than that in [6].

Proof of Theorem 1.1(i). Let $x\in G$; we must show that $x=\prod_{A\subset\langle n\rangle} x_A$ for unique $x_A\in M_A$. For uniqueness, suppose $x=\prod_{A\subset\langle n\rangle} x_A$ with $x_A\in M_A$. Then

$$\left(\prod_{j\notin A} s_j\right)x = \prod_{B\subset A} x_B \tag{12}$$

(where $\prod s_j$ is a composite), which can be solved for x_A in terms of x and the x_B for which B is a proper subset of A. Inductively, therefore, x determines the x_A.

For existence, define elements x_A of G inductively by (12). Putting $A=\langle n\rangle$ gives $x=\prod_{B\subset\langle n\rangle} x_B$, so it remains to show that $x_A\in M_A$. We use induction on the size of A. For $i\in A$, applying s_i to (12) and then using (12) with A replaced by $A\setminus\{i\}$ gives

$$\prod_{B\subset A} s_ix_B = \left(\prod_{j\notin A\setminus\{i\}} s_j\right)x = \prod_{B\subset A\setminus\{i\}} x_B.$$

Now the inductive hypothesis tells us that if B is a proper subset of A, then $s_ix_B=x_B$ for $B\subset A\setminus\{i\}$, and $s_ix_B=1$ otherwise. It follows that $s_ix_A=1$, so $x_A\in\operatorname{Ker} s_i$. For $i\notin A$ the inductive hypothesis and (12) immediately give $x_A\in\operatorname{Im} s_i$. So

$$x_A\in \bigcap_{i\in A}\operatorname{Ker} s_i \cap \bigcap_{i\notin A}\operatorname{Im} s_i = M_A,$$

as required. This completes the proof of Theorem 1.1(i). $\square$

It is straightforward to verify that (i) implies (ii) in Theorem 1.2. To prove the converse, and to prove Theorem 1.1(ii), we use the following lemma:

Lemma 1.4. *Suppose that (M_A, h) is a family of groups and functions satisfying* (5)–(11). *Let Φ be a family of subsets of $\langle n\rangle$ closed under union and let*

$$P_\Phi = \left(\mathop{*}_{A\in\Phi} M_A\right)/(h(a,b)=[a,b] \text{ for } a\in M_A,\ b\in M_B \text{ with } A,B\in\Phi).$$

Then for some total ordering of Φ, the multiplication function

$$(x_A) \mapsto \prod_A x_A : \prod_{A \in \Phi} M_A \to P_\Phi$$

is bijective.

Proof of Theorem 1.1(ii). Let P be P_Ω, where Ω is the family of all subsets of $\langle n \rangle$. Since (i) implies (ii) in Theorem 1.2, one can apply Lemma 1.4 to (M_A, h). Then Lemma 1.4 and Theorem 1.1(i) combine to show that the evident homomorphism $P \to G$ is bijective, which is what is required. □

Proof of Theorem 1.2. (ii) ⇒ (i). Let G be P_Ω, where Ω is the family of all subsets of $\langle n \rangle$. We can define commuting projections s_i on G as follows: if $a \in M_A$, then

$$s_i a = \begin{cases} 1, & i \in A, \\ a, & i \notin A. \end{cases}$$

The canonical homomorphisms $M_A \to G$ then restrict to give homomorphisms

$$M_A \to \bigcap_{i \in A} \operatorname{Ker} s_i \cap \bigcap_{i \notin A} \operatorname{Im} s_i .$$

By Lemma 1.4 and Theorem 1.1(ii) these must be isomorphisms. This completes the proof. □

It remains to prove Lemma 1.4. We use induction on the size of Φ, the case $\Phi = \emptyset$ being trivial. If Φ is not empty, then let A be a minimal member of Φ and let $\Psi = \Phi \setminus \{A\}$. The inductive hypothesis applies to Ψ, so it suffices to prove that P_Φ is a semi-direct product of M_A and P_Ψ. Now P_Φ is obtained from $M_A * P_\Psi$ by imposing the relations

$$h(b,c) = [b,c] \quad \text{for } b \in M_B,\ c \in M_C \text{ with } B, C \in \Phi, \text{ where } B = A \text{ or } C = A.$$

The relations with $B = C = A$ already hold in M_A, by (5), and the relations with $B = A$ are equivalent to those with $C = A$, by (6). So the only relations needed are $h(a,b) = [a,b]$ for $a \in M_A$, $b \in M_B$, $B \in \Psi$. Now $h(a,b) = [a,b]$ is equivalent to $h(a,b)b = aba^{-1}$, so it suffices to show that the functions

$$(a,b) \mapsto h(a,b)b : M_A \times M_B \to P_\Psi, \quad B \in \Psi$$

determine an action of M_A on P_Ψ.

To do this, we note that for fixed $a \in M_A$ the function $b \mapsto h(a,b)b : M_B \to P_\Psi$ is a homomorphism by (9) (the relations of P_Ψ yield ${}^b h(a,b') = bh(a,b')b^{-1}$ in P_Ψ for $b, b' \in M_B$). These homomorphisms determine an endomorphism $\gamma(a)$ of P_Ψ because of the following lemma, which will be proved later.

Lemma 1.5. *For $a \in M_A$, $b \in M_B$, $c \in M_C$ with $B, C \in \Psi$,*

$$h(a, h(b,c))h(b,c) = [h(a,b)b, h(a,c)c]$$

in P_Ψ.

Also, the function $\gamma : M_A \to \operatorname{End} P_\Psi$ is actually a homomorphism into $\operatorname{Aut} P_\Psi$, i.e. an action of M_A on P_Ψ, because of (7) and (8). This completes the proof of Lemma 1.4 assuming Lemma 1.5. □

Proof of Lemma 1.5. We shall need three identities in P_Ψ. First, since

$$e \mapsto h(a,e)e : M_E \to P_\Psi$$

is a homomorphism for $E \in \Psi$, by (9),

$$(h(a,e)e)^{-1} = h(a,e^{-1})e^{-1} \quad \text{for } e \in M_E,\ E \in \Psi. \tag{13}$$

Consequently,

$$eh(e^{-1},a) = h(a,e)e \quad \text{for } e \in M_E,\ E \in \Psi, \tag{14}$$

for (6) gives $eh(e^{-1},a) = eh(a,e^{-1})^{-1} = (h(a,e^{-1})e^{-1})^{-1}$, which is equal to $h(a,e)e$ by (13). Finally,

$$bh(a,d)d = h(a,b)b \quad \text{where } d = h(a^{-1},b^{-1}), \tag{15}$$

for

$$\begin{aligned} bh(a,d)d &= b^a d \\ &= b^a h(a^{-1},b^{-1}) \\ &= bh(aa^{-1},b^{-1})h(a,b^{-1})^{-1} \quad \text{by (8)} \\ &= bh(b^{-1},a) \quad \text{by (7) and (6)} \\ &= h(a,b)b \quad \text{by (14).} \end{aligned}$$

We next prove Lemma 1.5 in three special cases.

Case 1: $A \subset B$ and $A \subset C$. Here Lemma 1.5 reduces to (11).

Case 2: $A \subset C$ and B arbitrary. This follows from Case 1:

$$\begin{aligned} &h(a,h(b,c))h(b,c) = \\ &\quad = h(h(b,c),a)^{-1}bcb^{-1}c^{-1} \quad \text{by (6) and the relations of } P_\Psi \\ &\quad = b({}^{b^{-1}}h(h(b,c),a))^{-1}cb^{-1}c^{-1} \\ &\quad = b^a h(h(a^{-1},b^{-1}),c)^c h(h(c^{-1},a),b^{-1})cb^{-1}c^{-1} \quad \text{by (10)} \\ &\quad = b^a h(d,c)ch(c^{-1},a)b^{-1}h(c^{-1},a)^{-1}c^{-1}, \\ &\qquad \text{where } d = h(a^{-1},b^{-1}) \text{ as in (15),} \\ &\quad = bh(a,h(d,c))h(d,c)h(a,c)cb^{-1}h(a,c^{-1})c^{-1} \quad \text{by (14) and (6)} \\ &\quad = b[h(a,d)d, h(a,c)c]h(a,c)cb^{-1}h(a,c^{-1})c^{-1} \\ &\qquad \text{by Case 1, since } d \in M_{A\cup B} \text{ and } A \subset A \cup B, \end{aligned}$$

$$= [bh(a,d)d, h(a,c)c] \quad \text{by (13)}$$

$$= [h(a,b)b, h(a,c)c] \quad \text{by (15)}.$$

Case 3: $A \subset B$ and C arbitrary. This follows from Case 2 by inverting both sides; note from (13) and (6) that the inverse of $h(a,h(b,c))h(b,c)$ is $h(a,h(c,b))h(c,b)$.

Finally Lemma 1.5 in general follows from Case 3 in the same way as Case 2 follows from Case 1. This completes the proof. $\square$

2. r-universal crossed n-cubes

In this section we study the crossed n-cubes of the kinds which arise in the applications in the next section.

For an integer r a crossed n-cube (M_A) will be called *r-universal* if it is 'freely generated by the M_A with $|A| \le r$'. Formally, this means that if (N_A) is a crossed n-cube such that

$$N_A = M_A \text{ for } |A| \le r,$$

$$\mu_i \text{ on } N_A \text{ agrees with } \mu_i \text{ on } M_A \text{ for } |A| \le r,$$

$$h \text{ on } N_A \times N_B \text{ agrees with } h \text{ on } M_A \times M_B \text{ for } |A \cup B| \le r,$$

then there is a unique morphism $(M_A) \to (N_A)$ which is the identity on M_A for $|A| \le r$. Evidently the universal cat^n-groups of [4] correspond to $(n-1)$-universal crossed n-cubes under the equivalence of Theorem 1.3.

In [4] there is a description of 1-universal crossed 2-cubes (M_A): in such a 2-cube, $M_{\langle 2 \rangle}$ is a non-abelian tensor product of $M_{\{1\}}$ and $M_{\{2\}}$, which reduces to the ordinary tensor product of $M_{\{1\}}^{\mathrm{ab}}$ and $M_{\{2\}}^{\mathrm{ab}}$ if the actions

$$(a,b) \mapsto h(\mu_i a, b)b : M_{\{i\}} \times M_{\{j\}} \to M_{\{j\}}, \quad i=1, j=2 \text{ or } i=2, j=1$$

are trivial. We shall generalise this example in various ways.

Given a crossed n-cube (M_A) and given $a \in M_A$, $A \subset \langle n \rangle$, we shall write

$$\mu a = \left(\prod_{i \in A} \mu_i \right) a \in M_\emptyset.$$

We note that as a consequence of (4),

$$^{a}b = {}^{\mu a}b \quad \text{for } a \in M_A,\ b \in M_B \text{ with } A \subset B. \tag{16}$$

Theorem 2.1. *Let (M_A) be a 1-universal crossed n-cube and let A be a non-empty subset of $\langle n \rangle$. For $\zeta : \{1, \ldots, |A|\} \to A$ a bijection, let H_ζ be the subgroup of M_A generated by the elements*

$$h(x_1, h(x_2, \ldots, h(x_{|A|-1}, x_{|A|}) \ldots)), \quad x_i \in M_{\{\zeta(i)\}}.$$

Then each subgroup H_ζ is normal, and for any fixed $k \in A$, M_A is generated by the

$(|A|-1)!$ subgroups H_ζ for which $\zeta(|A|)=k$.

(If A is a singleton and $\zeta:\{1\}\to A$ the unique bijection, then H_ζ is to be taken as M_A.)

Proof. Let $x=h(x_1,\dots,h(x_{|A|-1},x_{|A|})\dots)$ be a generator of a subgroup H_ζ, and let y be any element of M_A. By (16) and (11),

$$\begin{aligned}yxy^{-1}&={}^{y}x={}^{\mu y}x\\&={}^{\mu y}h(x_1,\dots,h(x_{|A|-1},x_{|A|})\dots)\\&=h({}^{\mu y}x_1,\dots,h({}^{\mu y}x_{|A|-1},{}^{\mu y}x_{|A|})\dots),\end{aligned}$$

which is again a generator of H_ζ. So H_ζ is normal. (This part of the proof does not use 1-universality.)

To prove that M_A is generated by the H_ζ with $\zeta(|A|)=k$, we use induction on $|A|$. If $|A|=1$, then the result is obvious, so assume $|A|\ge 2$. The argument of the last paragraph shows in general that ${}^{y}x$ has the same form as x for $x\in M_X$, $y\in M_Y$, $Y\subset X\subset\langle n\rangle$, so we can often ignore superscripts during the argument.

By 1-universality, M_A is generated by elements obtained by repeated application of the h and μ_i to the M_U ($|U|\le 1$). Because of (3) we may assume that the μ_i are all applied before the h, so in fact each generator can be obtained from the M_U ($|U|\le 1$) by iterated applications of the h. Since $h(u,v)$ and $h(v,u)$ are in M_U for $u\in M_U$, $v\in M_\emptyset$, we can absorb the elements from $M_\emptyset$, so each generator can be obtained from the $M_{\{i\}}$ by iterated applications of the h.

We now see that M_A is generated by elements of the form $h(u,v)$ ($u\in M_U$, $v\in M_V$, $U\cup V=A$, $U,V\neq\emptyset$). Because of (4) we need such elements only when $U\cap V=\emptyset$; because of (6) we need such elements only when $k\in V$. We claim further that we need such elements only when U is a singleton.

Indeed, if U is not a singleton, then by the inductive hypothesis M_U is generated by elements of the form $h(w,x)$ ($w\in M_W$, $x\in M_X$, $W\cup X=U$, $W\cap X=\emptyset$, $W,X\neq\emptyset$). Using (8), and neglecting superscripts as we may, we see that $h(u,v)$ is a product of elements of the form $h(h(w,x),v')$ ($w\in M_W$, $x\in M_X$, $v'\in M_V$, $W\cup X=U$, $W\cap X=\emptyset$, $W,X\neq\emptyset$). But by (10) and (6), $h(h(w,x),v')$ can be written in the form

$$h(h(x_1,v_1),w_1)^{-1}h(h(v_2,w_2),x_2)^{-1}=h(w_1,h(x_1,v_1))h(x_2,h(v_2,w_2))$$

($x_i\in M_X$, $v_i\in M_V$, $w_i\in M_W$) (again we can neglect superscripts). Thus, if $u\in M_U$, $v\in M_V$ with $U\cup V=A$, $U\cap V=\emptyset$, $U,V\neq\emptyset$, $k\in V$, U not a singleton, then $h(u,v)$ is a product of similar elements $h(u',v')$ ($u'\in M_{U'}$, $v'\in M_{V'}$) with U' smaller than U. Iterating this, we see that we need such generators $h(u,v)$ only for U a singleton.

We now see that M_A is generated by elements of the form $h(x,y)$ ($x\in M_{\{i\}}$, $y\in M_{A\setminus\{i\}}$) with $i\in A$, $i\neq k$. Applying the inductive hypothesis to $M_{A\setminus\{i\}}$ and using (9) we see that M_A is generated by the generators of the H_ζ for which $\zeta(|A|)=k$ (again we can ignore superscripts). This completes the proof. $\square$

Theorem 2.2. *Let (M_A) be a 1-universal crossed n-cube such that $\mu_i : M_{\{i\}} \to M_\emptyset$ is trivial for $i \in \langle n \rangle$. Then the M_A are abelian for $A \neq \emptyset$, the functions $h : M_A \times M_B \to M_{A \cup B}$ are biadditive for $A, B \neq \emptyset$, and for $A \neq \emptyset$ and any fixed $k \in A$ the $(|A|-1)!$ multiadditive maps*

$$(a_1, \ldots, a_{|A|}) \mapsto h(a_1, h(a_2, \ldots, h(a_{|A|-1}, a_{|A|}) \ldots)) : M_{\{\zeta(1)\}} \times \cdots \times M_{\{\zeta(|A|)\}} \to M_A$$

$(\zeta : \{1, \ldots, |A|\} \to A$ a bijection with $\zeta(|A|) = k)$ express M_A as the direct sum of $(|A|-1)!$ copies of $\bigotimes_{i \in A} M_{\{i\}}$. Also all the $\mu_i : M_A \to M_{A \setminus \{i\}}$ are trivial for $i \in A$, and the $h : M_A \times M_B \to M_{A \cup B}$ are trivial for $A \cap B \neq \emptyset$.

Proof. From Theorem 2.1, (3) and (7) the $\mu_i : M_A \to M_{A \setminus \{i\}}$ are trivial for $i \in A$. From (4) and (7) the $h : M_A \times M_B \to M_{A \cup B}$ are trivial for $A \cap B \neq \emptyset$. From (16) the action of M_A on M_B is trivial if $A \subset B$ and $A \neq \emptyset$; in particular putting $B = A$ shows that M_A is abelian for $A \neq \emptyset$. It now follows from (8) and (9) that the $h : M_A \times M_B \to M_{A \cup B}$ are biadditive for $A, B \neq \emptyset$. The result of Theorem 2.1 now gives an epimorphism

$$\theta : \bigoplus^{(|A|-1)!} \bigotimes_{i \in A} M_{\{i\}} \to M_A$$

for $A \neq \emptyset$. It remains to show that θ is injective.

We use the Lie-theoretic method of [9]. Let R be the free associative (non-commutative) ring generated by $\bigoplus_{i \in \langle n \rangle} M_{\{i\}}$. Construct a crossed n-cube (R_A) as follows. First, $R_\emptyset = M_\emptyset$, and for $A \neq \emptyset$, R_A is the additive subgroup of R generated by $|A|$-fold products with one factor from each $M_{\{i\}}$ $(i \in A)$. This means that

$$R_A = \bigoplus_{\zeta} M_{\{\zeta(1)\}} \otimes \cdots \otimes M_{\{\zeta(|A|)\}}$$

summed over all bijections $\zeta : \{1, \ldots, |A|\} \to A$. The $\mu_i : R_A \to R_{A \setminus \{i\}}$ $(i \in A)$ are to be trivial. For $A \cap B \neq \emptyset$ the $h : R_A \times R_B \to R_{A \cup B}$ are to be trivial. For $A \cap B = \emptyset$ and $A, B \neq \emptyset$ the $h : R_A \times R_B \to R_{A \cup B}$ are given by

$$h(a, b) = ab - ba.$$

The $h : R_A \times R_B \to R_{A \cup B}$ where one of A, B is empty and the other is not are specified by giving an action of $R_\emptyset = M_\emptyset$ on R_A $(A \neq \emptyset)$: we set

$${}^{v}(a_1 \cdots a_{|A|}) = ({}^{v}a_1) \cdots ({}^{v}a_{|A|})$$

$(v \in M_\emptyset$, $a_i \in M_{\{\zeta(i)\}}$, $\zeta : \{1, \ldots, |A|\} \to A$ a bijection) and extend the action additively. Finally $h : R_\emptyset \times R_\emptyset \to R_\emptyset$ is the (group) commutator.

It is straightforward to check that (R_A) is a crossed n-cube. By 1-universality there is a homomorphism $\phi : (M_A) \to (R_A)$ which is the identity on the $M_{\{i\}}$ and $M_\emptyset$. It is easy to check that $\phi\theta$ is the obvious inclusion

$$\bigoplus_{(\zeta : \zeta(|A|) = k)} \bigotimes_{i \in A} M_{\{i\}} \to \bigoplus_{\text{all } \zeta} \bigotimes_{i \in A} M_{\{i\}},$$

so θ is injective. This completes the proof. $\square$

In the remainder of this section we consider an alternative generalisation of the 1-universal crossed 2-cubes of [4].

Theorem 2.3. *Let (M_A) be an $(n-1)$-universal crossed n-cube. Then $M_{\langle n\rangle}$ has a presentation with generators $a\otimes b$ $(a\in M_A,\ b\in M_B,\ A\cup B=\langle n\rangle,\ A\cap B=\emptyset,\ A,B\neq\emptyset)$ and with relations: for $a,a'\in M_A$, $b\in M_B$, $c\in M_C$, $d\in M_D$ with $A,B,C,D\neq\emptyset$,*

$$a\otimes b=(b\otimes a)^{-1}\quad \text{if } A\cup B=\langle n\rangle,\ A\cap B=\emptyset, \tag{17}$$

$$aa'\otimes b=(aa'a^{-1}\otimes {}^{\mu a}b)(a\otimes b)\quad \text{if } A\cup B=\langle n\rangle,\ A\cap B=\emptyset, \tag{18}$$

$$({}^{a}h(a^{-1},b)\otimes {}^{\mu a}c)({}^{c}h(c^{-1},a)\otimes {}^{\mu c}b)({}^{b}h(b^{-1},c)\otimes {}^{\mu b}a)=1$$
$$\text{if } A\cup B\cup C=\langle n\rangle,\ A\cap B=A\cap C=B\cap C=\emptyset, \tag{19}$$

$$\mu_i a\otimes b=a\otimes\mu_i b\quad \text{if } A\cup B=\langle n\rangle,\ A\cap B=\{i\},\ A,B\neq\langle n\rangle, \tag{20}$$

$$(a\otimes b)(c\otimes d)(a\otimes b)^{-1}={}^{[\mu a,\mu b]}c\otimes {}^{[\mu a,\mu b]}d$$
$$\text{if } A\cup B=C\cup D=\langle n\rangle,\ A\cap B=C\cap D=\emptyset. \tag{21}$$

Proof. It is easy to check, using (16), that equations (17)–(21) are consequences of the defining equations (1)–(11) of a crossed n-cube if one identifies $a\otimes b$ with $h(a,b)$. So to prove the theorem it suffices to show that there is a crossed n-cube (N_A) which is the same as (M_A) except where it involves $N_{\langle n\rangle}$, in which $N_{\langle n\rangle}$ has the presentation described in the theorem, and in which $h(a,b)=a\otimes b$ for $a\in N_A$, $b\in N_B$ with $A\cup B=\langle n\rangle$, $A\cap B=\emptyset$, $A,B\neq\emptyset$.

To do this one has to construct homomorphisms $\mu_i:N_{\langle n\rangle}\to N_{\langle n\rangle\setminus\{i\}}$ for $1\leq i\leq n$, and one has to construct functions $h:N_A\times N_B\to N_{A\cup B}$ for $A\cup B=\langle n\rangle$. One can check that there are well-defined homomorphisms $\mu_i:N_{\langle n\rangle}\to N_{\langle n\rangle\setminus\{i\}}$ given by $\mu_i(a\otimes b)=h(\mu_i a,\mu_i b)$ for $a\otimes b$ a generator. For $A\cup B=\langle n\rangle$ with $A,B\neq\langle n\rangle$ and $a\in N_A$, $b\in N_B$ we set

$$h(a,b)=\Big(\prod_{i\in C}\mu_i\Big)a\otimes\Big(\prod_{j\in D}\mu_j\Big)b,$$

where $\{C,D\}$ is any partition of $A\cap B$; in view of (20) it does not matter which partition is chosen. Also $h:N_{\langle n\rangle}\times N_{\langle n\rangle}\to N_{\langle n\rangle}$ is of course the commutator. To get $h:N_A\times N_B\to N_{\langle n\rangle}$ where $A=\langle n\rangle$ or $B=\langle n\rangle$ but not both, one checks that there is a well-defined action of N_A $(A\neq\langle n\rangle)$ on $N_{\langle n\rangle}$ given on generators by

$${}^{a}(b\otimes c)={}^{\mu a}b\otimes {}^{\mu a}c,\quad a\in N_A$$

and then sets

$$h(a,u)=({}^{a}u)u^{-1},\qquad h(u,a)=u({}^{a}u)^{-1}$$

for $a \in N_A$, $u \in N_{\langle n \rangle}$, $A \neq \emptyset$. It can now be checked that (N_A) is a crossed n-cube, which completes the proof. Note that identities involving $h: N_A \times N_B \to N_{\langle n \rangle}$ with $A = \langle n \rangle$ or $B = \langle n \rangle$ need to be checked only on generators of $N_{\langle n \rangle}$, since h is defined in terms of an action. Also, identities involving $h: N_A \times N_B \to N_{\langle n \rangle}$ with $A, B \neq \langle n \rangle$ can usually be reduced to the case $A \cap B = \emptyset$ by use of (4). □

We conclude this section with two special cases.

Theorem 2.4. *Let (M_A) be an $(n-1)$-universal crossed n-cube such that $\mu_i : M_{\{i\}} \to M_\emptyset$ is trivial for $i \in \langle n \rangle$. Then the M_A are abelian for $A \neq \emptyset$, and $M_{\langle n \rangle}$ is the quotient of*

$$\bigoplus_{A \cup B = \langle n \rangle,\, A \cap B = \emptyset,\, A, B \neq \emptyset} (M_A \otimes M_B)$$

by the relations (written additively):

$$a \otimes b = -b \otimes a \quad \text{for } a \in M_A,\ b \in M_B,\ A \cup B = \langle n \rangle,\ A \cap B = \emptyset,\ A, B \neq \emptyset, \tag{22}$$

$$(h(a,b) \otimes c) + (h(c,a) \otimes b) + (h(b,c) \otimes a) = 0$$
$$\text{for } a \in M_A,\ b \in M_B,\ c \in M_C,\ A \cup B \cup C = \langle n \rangle,\ A \cap B = A \cap C = B \cap C = \emptyset,$$
$$A, B, C \neq \emptyset, \tag{23}$$

$$\mu_i a \otimes b = a \otimes \mu_i b$$
$$\text{for } a \in M_A,\ b \in M_B,\ A \cup B = \langle n \rangle,\ A \cap B = \{i\},\ A, B \neq \langle n \rangle. \tag{24}$$

Proof. As in the proof of Theorem 2.2 it follows from (16) that the action of M_A on M_B is trivial for $A \subset B$, $A \neq \emptyset$; in particular M_A is abelian for $A \neq \emptyset$. Of the relations (17)–(21), (18) now reduces in view of (17) to biadditivity, (21) becomes trivial, and the rest reduce to (22)–(24). □

The other special case of Theorem 2.3 is not used in this paper. It is however used in [4], where there is a reference to the following theorem.

Theorem 2.5. *Let G be a group and (M_A) an $(n-1)$-universal crossed n-cube ($n \geq 2$) such that $M_A = G$ for all $A \neq \langle n \rangle$ and such that $\mu_i : M_A \to M_{A \setminus \{i\}}$ is the identity map for all $A \neq \langle n \rangle$. Then $M_{\langle n \rangle}$ has a presentation with generators $a \otimes b$ ($a, b \in G$) and relations: for all $a, a', b, b' \in G$,*

$$aa' \otimes b = (aa'a^{-1} \otimes aba^{-1})(a \otimes b), \tag{25}$$

$$a \otimes bb' = (a \otimes b)(bab^{-1} \otimes bb'b^{-1}), \tag{26}$$

and for $n \geq 3$ also

$$a \otimes b = (b \otimes a)^{-1}. \tag{27}$$

Proof. For $n=2$ the result is given in [3], so suppose $n \geq 3$. Let N be the group with the presentation given in Theorem 2.5. For $a,b,c \in G$ write ${}^a b = aba^{-1}$ in G and ${}^a(b \otimes c) = {}^a b \otimes {}^a c$ in N; clearly this defines an action of G on N. One now sees that $(b,c) \mapsto b \otimes c : G \times G \to N$ is a crossed pairing in the sense of [3]. From [3] the following identities hold in N:

$$(a \otimes b)(c \otimes d)(a \otimes b)^{-1} = {}^{[a,b]}c \otimes {}^{[a,b]}d, \tag{28}$$

$$[a,b] \otimes c = (a \otimes b)({}^c b \otimes {}^c a) \tag{29}$$

for $a,b,c,d \in G$. We have also

$$({}^a[a^{-1},b] \otimes {}^a c)({}^c[c^{-1},a] \otimes {}^c b)({}^b[b^{-1},c] \otimes {}^b a) = 1 \tag{30}$$

for $a,b,c \in G$, since

$$\begin{aligned}
&({}^a[a^{-1},b] \otimes {}^a c)({}^c[c^{-1},a] \otimes {}^c b)({}^b[b^{-1},c] \otimes {}^b a) = \\
&\quad = ({}^a[a^{-1},b] \otimes {}^a c)([a,c] \otimes [c,b]b)([c,b] \otimes {}^b a) \\
&\quad = ({}^a[a^{-1},b] \otimes {}^a c)(a \otimes c)({}^{[c,b]b}c \otimes {}^{[c,b]b}a)([c,b] \otimes {}^b a) \quad \text{by (29)} \\
&\quad = (a[a^{-1},b] \otimes c)([c,b]\,{}^b c \otimes {}^b a) \quad \text{by (25)} \\
&\quad = ({}^b a \otimes c)(c \otimes {}^b a) = 1 \quad \text{by (27)}.
\end{aligned}$$

In $M_{\langle n \rangle}$ as presented in Theorem 2.3 we have elements $a \otimes b$ $(a,b \in G)$ for each ordered partition of $\langle n \rangle$ into non-empty subsets A, B. But (20) shows that we get the same element of $M_{\langle n \rangle}$ for each choice of partition, since $n \geq 3$. It is now clear that $M_{\langle n \rangle} = N$; we use (28) and (30) in particular. This completes the proof. □

3. Applications to n-cubes of maps and $(n+1)$-ads

In this section we apply the theory of crossed n-cubes to n-cubes of maps and $(n+1)$-ads. We describe the fundamental crossed n-cube of an n-cube of maps, consider higher homotopy groups, and relate the theory to $(n+1)$-ads. Finally we use the results of Section 2 to describe the homotopy groups of particular $(n+1)$-ads. The methods generalise those applied to squares of maps and triads by Brown and Loday [2–4].

We shall parametrise an n-cube of maps as a commutative diagram X consisting of spaces X_A $(A \subset \langle n \rangle)$ and maps $X_A \to X_{A \cup \{i\}}$ $(A \subset \langle n \rangle,\ i \in \langle n \rangle,\ i \notin A)$. For instance, if $n=2$ we get a commutative square

$$\begin{tikzcd}
X_\emptyset \arrow[r] \arrow[d] & X_{\{1\}} \arrow[d] \\
X_{\{2\}} \arrow[r] & X_{\langle 2 \rangle}
\end{tikzcd}$$

As in [3, 4] an n-cube of maps X yields an n-cube of fibrations $\bar{X}$. This we shall parametrise as a commutative diagram consisting of spaces $\bar{X}_{A,B}$ (A, B disjoint subsets of $\langle n\rangle$) and fibrations

$$\bar{X}_{A\cup\{i\},B} \to \bar{X}_{A,B} \to \bar{X}_{A,B\cup\{i\}}, \quad A\cap B=\emptyset,\ i\in\langle n\rangle\setminus(A\cup B).$$

This contains an n-cube of maps $(\bar{X}_{\emptyset,*})$ homotopy equivalent to X (i.e. there is a morphism $X\to(\bar{X}_{\emptyset,*})$ consisting of homotopy equivalences $X_B \xrightarrow{\simeq} \bar{X}_{\emptyset,B}$). For instance, if $n=2$ we get a commutative diagram

$$\begin{tikzcd}
\bar{X}_{\langle 2\rangle,\emptyset} \arrow[r] \arrow[d] & \bar{X}_{\{2\},\emptyset} \arrow[r] \arrow[d] & \bar{X}_{\{2\},\{1\}} \arrow[d] \\
\bar{X}_{\{1\},\emptyset} \arrow[r] \arrow[d] & \bar{X}_{\emptyset,\emptyset} \arrow[r] \arrow[d] & \bar{X}_{\emptyset,\{1\}} \arrow[d] \\
\bar{X}_{\{1\},\{2\}} \arrow[r] & \bar{X}_{\emptyset,\{2\}} \arrow[r] & \bar{X}_{\emptyset,\langle 2\rangle}
\end{tikzcd}$$

in which the rows and columns are fibrations and the square at the bottom right is homotopy equivalent to X. We note an obvious fact.

Proposition 3.1. *If $\bar{X}$ is an n-cube of fibrations, then $\bar{X}_{A,B}$ is the intersection over $i\in A$ of the fibres of $\bar{X}_{\emptyset,B}\to\bar{X}_{\emptyset,B\cup\{i\}}$.*

Thus an n-cube of maps X yields an n-cube of fibrations $\bar{X}$. As in [3, 4] this in turn yields a fundamental cat^n-group $\Pi_1 X$, which by Theorem 1.3 we may regard as a crossed n-cube. When we do so we find that

$$(\Pi_1 X)_A = \pi_1(\bar{X}_{A,\emptyset}) \quad \text{for } A\subset\langle n\rangle$$

and the $\mu_i : (\Pi_1 X)_A \to (\Pi_1 X)_{A\setminus\{i\}}$ ($i\in A$) are induced by the maps in the n-cube of fibrations.

Next we bring higher homotopy groups into the picture. Let $q=(q_1,\dots,q_n)$ be an n-tuple of positive integers, fixed in what follows. For A a subset of $\langle n\rangle$, write

$$A^* = \{(i,r):\ i\in A,\ r \text{ is an integer},\ 1\le r\le q_i\}$$

(thus $|A^*| = \sum_{i\in A} q_i$). Given an n-cube of maps X we shall construct a $(q_1+\dots+q_n)$-cube of maps X_q indexed by the subsets of $\langle n\rangle^*$ as follows:

$$X_{q,A} = X_{A'}, \quad \text{where } A'=\{i\in\langle n\rangle:\ \{i\}^*\subset A\},$$

each map in X_q is either an identity map or a map in X.

For instance, suppose that $n=1$ and $q_1=2$. Then X_q is the square

$$\begin{tikzcd} X_\emptyset \arrow[r, "="] \arrow[d, "="'] & X_\emptyset \arrow[d] \\ X_\emptyset \arrow[r] & X_{\langle 1\rangle} \end{tikzcd}$$

As before, the $(q_1+\cdots+q_n)$-cube of maps X_q gives rise to a $(q_1+\cdots+q_n)$-cube of fibrations $\bar{X}_q$ and to a fundamental crossed $(q_1+\cdots+q_n)$-cube $\Pi_1 X_q$. This contains a crossed n-cube $\Pi_q X$ given by

$$(\pi_q X)_A=(\Pi_1 X_q)_{A^*}=\pi_1(\bar{X}_{q,A^*,\emptyset}) \quad \text{for } A\subset\langle n\rangle,$$

and we take this as the higher crossed n-cube of X.

Suppose for instance that $n=1$, $q_1=2$. Then the spaces of $\bar{X}_q$ are given up to homotopy type as

$$\begin{tikzcd} \Omega\bar{X}_{\langle 1\rangle,\emptyset} \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & \bar{X}_{\langle 1\rangle,\emptyset} \arrow[d] \\ * \arrow[r] \arrow[d] & \bar{X}_{\emptyset,\emptyset} \arrow[r] \arrow[d] & \bar{X}_{\emptyset,\emptyset} \arrow[d] \\ \bar{X}_{\langle 1\rangle,\emptyset} \arrow[r] & \bar{X}_{\emptyset,\emptyset} \arrow[r] & \bar{X}_{\emptyset,\langle 1\rangle} \end{tikzcd}$$

$$(\Pi_q X)_\emptyset=\pi_1(\bar{X}_{\emptyset,\emptyset}), \qquad (\Pi_q X)_{\langle 1\rangle}=\pi_1(\Omega\bar{X}_{\langle 1\rangle,\emptyset})=\pi_2(\bar{X}_{\langle 1\rangle,\emptyset}),$$

and $\mu_1:(\Pi_q X)_{\langle 1\rangle}\to(\Pi_q X)_\emptyset$ is trivial.

We can easily compute the spaces in $\bar{X}_q$ in general, though the results are notationally complicated. Call a sextuple (A,B,C,A',A'',C') *canonical* if

(i) A,B,C are disjoint subsets of $\langle n\rangle$,

(ii) A',A'',C' are disjoint subsets of $\langle n\rangle^*$,

(iii) $p(A')=A$, $p(A'')\subset A$, $p(C')=C$, where $p:\langle n\rangle^*\to\langle n\rangle$ is projection onto the first factor,

(iv) for every $i\in C$ there exists r such that $(i,r)\in\langle n\rangle^*$, $(i,r)\notin C'$.

The point of this is that any pair of disjoint subsets of $\langle n\rangle^*$ can be written in the form

$$(A',A''\cup B^*\cup C')$$

for a unique canonical sextuple (A,B,C,A',A'',C'). For instance, if $n=6$, $q=$

$(2,2,2,2,2,2)$,

$$D=\{(1,1),(2,1),(3,1),(3,2)\}, \qquad E=\{(2,2),(4,1),(4,2),(5,1)\},$$

then $(D,E)=(A',A''\cup B^*\cup C')$ where $A=\{1,2,3\}$, $B=\{4\}$, $C=\{5\}$, $A'=D$, $A''=\{(2,2)\}$, $C'=\{(5,1)\}$.

Proposition 3.2. *Let X be an n-cube of maps and (A,B,C,A',A'',C') be a canonical sextuple. Then*

$$\bar{X}_{q,A',A''\cup B^*\cup C'}\simeq\begin{cases}\Omega^{|A'|-|A|}\bar{X}_{A,B} & \textit{if } A'\cup A''=A^*,\\ * & \textit{otherwise.}\end{cases}$$

Proof. This is immediate from the definition if A', hence also A and A'', are empty. Otherwise $\bar{X}_{q,A',A''\cup B^*\cup C'}$ is the fibre of

$$\alpha:\bar{X}_{q,A'\setminus\{(i,r)\},A''\cup B^*\cup C'}\to\bar{X}_{q,A'\setminus\{(i,r)\},A''\cup\{(i,r)\}\cup B^*\cup C'}$$

for any $(i,r)\in A'$. If $A'\cup A''\neq A^*$, then Proposition 3.1 shows that α is a map between intersections of fibres of equivalent maps, so $\bar{X}_{q,A',A''\cup B^*\cup C'}$ is contractible. If $A'\cup A''=A^*$ and $|A'|=|A|$, then $\bar{X}_{q,A',A''\cup B^*\cup C'}\simeq\bar{X}_{A,B}$ by definition and by Proposition 3.1. Finally, for $A'\cup A''=A^*$ and $|A'|>|A|$ the result follows by induction on $|A'|-|A|$, using the fibration α for an (i,r) so chosen that there exists (i,s) in A'' with $s\neq r$. $\square$

Corollary 3.3. *Let X be an n-cube of maps. Then*

(i) *Π_qX contains all the non-trivial groups in Π_1X_q,*

(ii) *if $i\in A\subset\langle n\rangle$ and $q_i>1$, then*

$$\mu_i:(\Pi_qX)_A\to(\Pi_qX)_{A\setminus\{i\}}$$

is trivial.

Proof. (i) By Proposition 3.2, $(\Pi_1X_q)_{A'}=\pi_1(\bar{X}_{q,A',\emptyset})$ can be non-trivial only if $A'=A^*$ for some $A\subset\langle n\rangle$, in which case $(\Pi_1X_q)_{A'}$ is $(\Pi_qX)_A$.

(ii) Here μ_i is induced by $\bar{X}_{q,A^*,\emptyset}\to\bar{X}_{q,A^*\setminus\{i\}^*,\emptyset}$, which factors through a contractible space $\bar{X}_{q,A',\emptyset}$ for any set A' properly between A^* and $A^*\setminus\{i\}^*$. $\square$

Next we relate the theory to $(n+1)$-ads. For our purposes an $(n+1)$-ad will be an n-cube of maps X such that all the maps are inclusions and $X_{A\cap B}=X_A\cap X_B$ for $A,B\subset\langle n\rangle$. We shall write $X_i=X_{\langle n\rangle\setminus\{i\}}$ for $1\leq i\leq n$; we then have

$$X_A=\bigcap_{i\notin A}X_i \quad \text{for } A\neq\langle n\rangle.$$

Clearly an $(n+1)$-ad in this sense is determined by a space $X_{\langle n\rangle}$ together with n subspaces $X_1,\dots,X_n$ all containing the base-point. So our $(n+1)$-ad X is equivalent to the classical $(n+1)$-ad $(X_{\langle n\rangle};X_1,\dots,X_n)$. One can also check that the homotopy groups of the spaces $X_{A,B}$ in the corresponding n-cube of fibrations are $(|A|+1)$-ad homotopy groups (see [1]). The precise statement is as follows.

Proposition 3.4. *If X is an $(n+1)$-ad, then*

$$\pi_m(\bar{X}_{A,B})=\pi_{m+|A|}(X_{A\cup B};X_{A\cup B}\cap X_i\colon i\in A)$$

for $A,B\subset\langle n\rangle$, $A\cap B=\emptyset$.

We now apply Proposition 3.2 to obtain

Proposition 3.5. *Let X be an $(n+1)$-ad and (A,B,C,A',A'',C') be a canonical sextuple. Then*

$$\pi_m(\bar{X}_{q,A',A''\cup B^*\cup C'})$$
$$=\begin{cases}\pi_{m+|A'|}(X_{A\cup B};X_{A\cup B}\cap X_i\colon i\in A) & \text{if } A'\cup A''=A^*,\\ 0 & \text{otherwise.}\end{cases}$$

In particular

$$(\Pi_qX)_A=\pi_{|A^*|+1}(X_A;X_A\cap X_i\colon i\in A)$$

for $A\subset\langle n\rangle$.

(Recall that $|A^*|=\sum_{i\in A}q_i$.)

The arguments of [4] show that if X is an $(n+1)$-ad, then the h-functions in Π_qX are generalised Whitehead products, up to sign.

We now come to the applications. Following [3] we call an n-cube of maps X *connected* if each space $\bar{X}_{A,B}$ in the corresponding n-cube of fibrations is path-connected. We shall call the n-cube of maps *q-connected* if each $\bar{X}_{A,B}$ is $(\sum_{i\in A}q_i)$-connected. In view of Proposition 3.4, an $(n+1)$-ad X is q-connected if and only if for all disjoint $A,B\subset\langle n\rangle$ we have

$$\pi_m(X_{A\cup B};X_{A\cup B}\cap X_i\colon i\in A)=0\quad\text{for } m\le\sum_{i\in A}q_i.$$

The following is immediate from Proposition 3.2:

Proposition 3.6. *An n-cube of maps X is q-connected if and only if X_q is connected.*

Brown and Loday [4] have shown that if X is an $(n+1)$-ad such that $X_1,\dots,X_n$ is an open covering of $X_{\langle n\rangle}$ and the n-ads $(X_i;X_i\cap X_j\colon j\neq i)$ are connected for $1\le i\le n$, then X is connected and Π_1X is $(n-1)$-universal. (Recall that their universal cat^n-groups correspond to our $(n-1)$-universal crossed n-cubes under the equivalence of Theorem 1.3.) In view of Proposition 3.4 the connectedness hypothesis of their theorem is equivalent to $\bar{X}_{A,B}$ being connected for $A\cup B\neq\langle n\rangle$. The theorem extends to higher homotopy groups as follows.

Theorem 3.7. *Let X be an $(n+1)$-ad such that $X_1, \ldots, X_n$ is an open cover of $X_{\langle n\rangle}$. Suppose that the n-ad*

$$(X_i; X_i \cap X_j : j \neq i)$$

is $(q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n)$-connected for $1 \leq i \leq n$. Let

$$Q = q_1 + \cdots + q_n.$$

Then X is q-connected and $\Pi_q X$ is $(n-1)$-universal. Hence

$$\pi_m(X_{\langle n\rangle}; X_1, \ldots, X_n) = 0 \quad \textit{for } m \leq Q$$

and

$$\pi_{Q+1}(X_{\langle n\rangle}; X_1, \ldots, X_n) = (\Pi_q X)_{\langle n\rangle}$$

has the presentation of Theorem 2.3 in terms of the

$$\pi_{|A^*|+1}(X_A; X_A \cap X_i : i \in A) = (\Pi_q X)_A.$$

If $q_i > 1$ for all i, then $\pi_{Q+1}(X_{\langle n\rangle}; X_1, \ldots, X_n)$ has the presentation of Theorem 2.4.

Proof. It is straightforward to check from the definition of X_q that X_q is a $(Q+1)$-ad with the $X_{q,\langle n\rangle^* \setminus \{(i,r)\}}$ $((i,r) \in \langle n\rangle^*)$ forming an open cover of $X_{q,\langle n\rangle^*}$. If (A, B, C, A', A'', C') is a canonical sextuple such that $A' \cup (A'' \cup B^* \cup C') \neq \langle n\rangle^*$, then $A' \cup A'' \neq A^*$ or $A \cup B \neq \langle n\rangle$. Either way it follows from Proposition 3.2 that $\bar{X}_{q,A',A''\cup B^*\cup C'}$ is path-connected; note that

$$|A'| - |A| \leq |A^*| - |A| = \sum_{i \in A} (q_i - 1).$$

So Brown and Loday's theorem tell us that X_q is connected and $\Pi_1 X_q$ is $(|\langle n\rangle^*| - 1)$-universal. By Proposition 3.6, X is q-connected, which includes the assertion that $\pi_m(X_{\langle n\rangle}; X_1, \ldots, X_n) = 0$ for $m \leq Q$, by Proposition 3.5. Also the $(|\langle n\rangle^*| - 1)$-universality of $\Pi_1 X_q$ gives the $(n-1)$-universality of $\Pi_q X$ by Corollary 3.3(i). So $(\Pi_q X)_{\langle n\rangle}$ has the presentation of Theorem 2.3. If $q_i > 1$ for all i, then $\mu_i : (\Pi_q X)_{\{i\}} \to (\Pi_q X)_\emptyset$ is trivial for $1 \leq i \leq n$ by Corollary 3.3(ii), so $(\Pi_q X)_{\langle n\rangle}$ has the presentation of Theorem 2.4. □

Our final result is an iterated application of Theorem 3.7.

Theorem 3.8. *Let X be an $(n+1)$-ad such that $X_1, \ldots, X_n$ are open subsets of $X_{\langle n\rangle}$ and every point of $X_{\langle n\rangle}$ is in at least $n-1$ of the X_i. Suppose that $\bigcap_{i \in \langle n\rangle} X_i$ is path-connected and that $(X_{\{i\}}; \bigcap_{j \in \langle n\rangle} X_j)$ is q_i-connected for $1 \leq i \leq n$. Let*

$$Q = q_1 + \cdots + q_n.$$

Then X is q-connected and $\Pi_q X$ is 1-universal. Hence

$$\pi_m(X_{\langle n\rangle}; X_1, \ldots, X_n) = 0 \quad \text{for } m \leq Q$$

and $\pi_{Q+1}(X_{\langle n\rangle}; X_1, \ldots, X_n)$ is generated by iterated Whitehead products

$$[a_1, [a_2, \ldots, [a_{n-1}, a_n] \ldots]]$$

($a_i \in \pi_{q_{\zeta(i)}+1}(X_{\{\zeta(i)\}}, \bigcap_{j\in\langle n\rangle} X_j)$ for a permutation ζ of $\langle n\rangle$ with $\zeta(n) = n$). If $q_i > 1$ for all i, then these Whitehead products yield an isomorphism

$$\pi_{Q+1}(X_{\langle n\rangle}; X_1, \ldots, X_n) \cong \bigoplus^{(n-1)!} \bigotimes_{i\in\langle n\rangle} \pi_{q_i+1}\left(X_{\{i\}}, \bigcap_{j\in\langle n\rangle} X_j\right).$$

This generalises the theorem of Barratt and Whitehead [1, 5.3], which requires $q_i \geq 2$ and $\bigcap_{j\in\langle n\rangle} X_j$ to be 1-connected.

Proof. The condition that each point of $X_{\langle n\rangle}$ should be in at least $(n-1)$ of the X_i implies that for each $A \subset \langle n\rangle$, the $X_A \cap X_i$ $(i \in A)$ form an open covering of X_A. So an inductive application of Theorem 3.7 shows that X is q-connected and that for each $A \subset \langle n\rangle$ with $|A| \geq 2$ the crossed $|A|$-cube $((\Pi_q X)_B : B \subset A)$ is $(|A|-1)$-universal. It follows by induction on the size of A that each crossed $|A|$-cube $((\Pi_q X)_B : B \subset A)$ $(A \subset \langle n\rangle)$ is 1-universal; in particular $\Pi_q X$ is 1-universal. The rest now follows from Theorems 2.1 and 2.2 as in the proof of Theorem 3.7. □

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